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The rook partition algebra

Cheryl Grood

Department of Mathematics and Statistics, Swarthmore College, Swarthmore, PA 19081, USA

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Abstract

The rook partition algebra $RP_k(x)$ is a generically semisimple algebra that arises from looking at what commutes with the action of the symmetric group S_n on $U^{\otimes k}$, where U is the direct sum of the natural representation and the trivial representation of S_n. We give a combinatorial description of this algebra, construct its irreducible representations, and exhibit a Murnaghan–Nakayama formula to compute certain character values.

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1. Introduction

Let G be a group and let M be a finite-dimensional G-module. The representations of G and its centralizer algebra $End_G(M)$ are fundamentally related. Many interesting algebras have arisen from determining $End_G(M)$ when G is one of Weyl's classical groups. The most well-known example of this phenomenon is a result from the beginning of the 20th century known as Schur–Weyl duality. Let $V = \mathbb{C}^n$, and let $GL(n)$ denote the group of $n \times n$ invertible matrices over $\mathbb{C}$. Then $GL(n)$ acts on V by matrix multiplication and on the k-fold tensor product $V^{\otimes k}$ diagonally:

$$g \cdot (v_1 \otimes \cdots \otimes v_k) = g \cdot v_1 \otimes \cdots \otimes g \cdot v_k.$$

Note that S_k, the symmetric group on k letters, also acts on $V^{\otimes k}$ by permuting the factors of a k-tensor; this action clearly commutes with the action of $GL(n)$. Therefore, $End_{GL(n)}(V^{\otimes k})$

E-mail address: cgrood1@swarthmore.edu.

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contains a homomorphic image of the group algebra $\mathbb{C}S_k$. In fact, the following key theorem showed that there is nothing more that commutes with the action of $GL(n)$.

Theorem 1 (*Schur–Weyl duality; Weyl [19,20,24]*). *$GL(n)$ and S_k generate full centralizers of each other on $V^{\otimes k}$. In particular, when $n \geqslant k$, $End_{GL(n)}(V^{\otimes k}) \cong \mathbb{C}S_k$.*

This theorem provides the connection between the representation theories (and related combinatorics) of the symmetric group and the general linear group.

1.1. Subgroups of $GL(n)$

If M is a G-module, and H is a subgroup of G, then M is an H-module by restriction. Since anything commuting with the action of G must also commute with the action of H, it follows that $End_G(M) \subseteq End_H(M)$.

In a 1937 paper, Brauer [3] investigated the centralizer algebras associated with subgroups of the general linear group—namely, the orthogonal groups $O(n, \mathbb{C})$, the special orthogonal groups $SO(n, \mathbb{C})$, and the symplectic groups $SP(2n, \mathbb{C})$—on tensor powers of their natural representations. Brauer determined the generators of these centralizer algebras and furthermore gave a combinatorial description of these algebras in terms of certain diagrams on $2k$ vertices, where k is the number of times the natural representation is tensored with itself. While there was not much further study of these algebras for several decades, in the late 1980s and 1990s others ([5,9,18,23], to name a few) began to look at these algebras with renewed interest. These centralizer algebras are now called *Brauer algebras* (when $G = O(n)$, $SO(2n + 1)$, or $SP(2n)$) and *even Brauer algebras* (when $G = SO(2n)$) and are denoted, respectively, $B_k(n)$, $B_k(2n + 1)$, $B_k(2n)$, and $D_k(2n)$.

One can view the symmetric group S_n as the subgroup of $GL(n)$ consisting of permutation matrices. The centralizer algebra $End_{S_n}(V^{\otimes k})$ is denoted $P_k(n)$ and was investigated in the early 1990s by Martin [12–14] and, later, independently arose in the work of Jones [10]. Because elements of $P_k(n)$ can be described in terms of diagrams that correspond to set partitions of $\{1, 2, \dots, 2k\}$, it is called the *partition algebra*.

Theorem 2 (*Schur–Weyl duality for the partition algebras; Jones [10], Martin [12]*). *S_n and $P_k(n)$ generate full centralizers of each other on $V^{\otimes k}$. In particular, when $n \geqslant k$, $End_{S_n}(V^{\otimes k}) \cong P_k(n)$.*

Since $S_n \subseteq O(n) \subseteq GL(n)$, one can realize the centralizer algebras $\mathbb{C}(S_k)$ and $B_k(n)$ as subalgebras of $P_k(n)$.

1.2. The rook monoid and some generalizations

The *rook monoid* R_k is the set of all $k \times k$ matrices that contain at most one entry of one in each column and row and zeroes elsewhere under matrix multiplication.[1] While

[1] The name "rook" stems from the correspondence between the matrices and the placement of non-attacking rooks on an $k \times k$ chessboard.

investigating representations of the rook monoid, L. Solomon realized that the rook monoid centralized the action of $GL(n)$ on a particular module.

Theorem 3 (*Schur–Weyl duality for the rook monoid; Solomon [21]*). *Let U be an $n+1$-dimensional space isomorphic to the direct sum of the natural representation V of $GL(n)$ and its trivial representation $\mathbb{C}$. Then $GL(n)$ and $\mathbb{C}R_k$ generate full centralizers of each other on $U^{\otimes k}$. In particular, when $n \geqslant k$, $End_{GL(n)}(U^{\otimes k}) \cong \mathbb{C}R_k$.*

The representation theory of the rook monoid, like the symmetric group, contains some beautiful combinatorics (see [4,6,21]). One might naturally wonder if, as before, interesting algebras would arise from determining the centralizers of subgroups of $GL(n)$ on $U^{\otimes k}$. Benkart and Halverson [1] have investigated the "rook Brauer algebras" $RB_k(n)$, centralizer algebras of the orthogonal groups on $U^{\otimes k}$. In this paper, we consider the subgroup S_n of $GL(n)$ and determine its centralizer algebra $End_{S_n}(U^{\otimes k})$ and a related diagram algebra, which we call the "rook partition algebra" $RP_k(n+1)$. This algebra was independently discovered by Martin [15,16] in a slightly different guise. Martin arrives at this algebra, denoted $P_k^1(n+1)$, by starting with $P_k(n+1)$, the centralizer algebra of S_{n+1} on $(\mathbb{C}^{n+1})^{\otimes k}$, and then looking at what centralizes a subgroup of S_{n+1} isomorphic to S_n. Since $\mathbb{C}^{n+1}$ is isomorphic to U when viewed as an S_n-module, $P_k^1(n+1) \cong RP_k(n+1)$. Inspired in part by the results of Martin and preliminary results from this paper, Halverson and Ram [8] give a unifying presentation of the structure and representation theory of the rook partition algebras (in their paper, referred to as the *partition algebras* $A_{k+\frac{1}{2}}(n)$) and the partition algebras $A_k(n)$.

We begin by explicitly deriving the centralizer algebra $End_{S_n}(U^{\otimes k})$ and defining a related diagram algebra. It should be noted that our diagram algebra presentation of $End_{S_n}(U^{\otimes k})$ differs from the one used in [8,15,16], and it has the advantage of containing S_k, $B_k(n+1)$, $P_k(n+1)$, R_k, and $RB_k(n+1)$ as diagram subalgebras. Next, we give an explicit construction of the irreducible representations of $RP_k(n+1)$. Finally, we present a recursive combinatorial formula to compute the character values of the irreducible representations.

2. The rook partition algebra

2.1. A centralizer algebra

Let $\{v_0\}$ be a basis for $\mathbb{C}$ and let $\{v_1, v_2, \ldots, v_n\}$ denote the standard basis of $V \cong \mathbb{C}^n$. As discussed in Section 1.2, for large enough n, the rook monoid is the centralizer algebra of $GL(n)$ on $U^{\otimes k}$, where $U = \mathbb{C} + V$. Note that the restricted action of S_n on U is given by

$$\sigma \cdot v_i = \begin{cases} v_0 & \text{if } i = 0, \\ v_{\sigma(i)} & \text{otherwise.} \end{cases}$$

To compute the centralizer algebra, we employ a technique used by Jones [10] in studying the partition algebras.

Let $I = (i_i, i_2, \ldots, i_k)$ be a k-tuple of elements from the set $\{0, 1, 2, \ldots, n\}$. For ease of notation, we will write v_I to denote the k-tensor $v_{i_1} \otimes \cdots \otimes v_{i_k}$. For $\sigma \in S_n$ we will use $\sigma(I)$ to represent the k-tuple $(\sigma(i_1), \ldots, \sigma(i_k))$, where σ fixes 0 and naturally permutes $1, 2, \ldots, n$. Let X be an endomorphism of $U^{\otimes k}$; we will write X_I^J as shorthand for $Xv_I|_{v_J}$, the coefficient of v_J in the product Xv_I.

Theorem 4. *Suppose X is an endomorphism of $U^{\otimes k}$. Then*

$$X \in End_{S_n}(U^{\otimes k}) \iff X_I^J = X_{\sigma(I)}^{\sigma(J)},$$

for all $\sigma \in S_n$ and for all k-tuples I, J.

Proof. An endomorphism X is in the centralizer $End_{S_n}(U^{\otimes k})$ if and only if $\sigma^{-1}X\sigma = X$ for all $\sigma \in S_n$. It suffices to determine under what conditions

$$(\sigma^{-1}X\sigma)v_I|_{v_J} = X_I^J$$

for all I, J, given an arbitrary element $\sigma \in S_n$. We have that $(\sigma^{-1}X\sigma)v_I = \sigma^{-1}Xv_{\sigma(I)}$. Observe that, in general

$$\sigma^{-1}Xv_{\sigma(I)}|_{v_{\sigma^{-1}(L)}} = X_{\sigma(I)}^L.$$

Hence, the coefficient of v_J in $\sigma^{-1}Xv_{\sigma(I)} = (\sigma^{-1}X\sigma)v_I$ is $X_{\sigma(I)}^{\sigma(J)}$. $\square$

2.2. Bases for $End_{S_n}(U^{\otimes k})$

Let $\Pi = \{1, 2, \ldots, 2k\}$ and let R be a subset of Π. For every such subset, there exist numerous set partitions P of $\Pi \setminus R$.

Definition 5. Let $\Lambda(\Pi) = \{(R, P) | R \subseteq \Pi,\ P \text{ is a set partition of } \Pi \setminus R\}$.

For each element (R, P) of $\Lambda(\Pi)$, we define an endomorphism $X(R, P)$ of $U^{\otimes k}$.

Definition 6.

$$X(R, P)_{(a_1,\ldots,a_k)}^{(a_{k+1},\ldots,a_{2k})} = \begin{cases} 1 & \text{if } a_i = 0 \iff i \in R \\ & \text{and } a_i = a_j, a_i \neq 0 \iff i, j \\ & \text{in same part of } P, \\ 0 & \text{otherwise,} \end{cases}$$

where $a_i \in \{0, 1, \ldots, n\}$ for all i.

Let us look at examples of such an endomorphism and its corresponding action on a given vector in $U^{\otimes k}$.

Example 7. Let $n = 10$ and $k = 4$. Suppose $R = \{4, 5\}$ and P is the set partition $\{\{1, 2, 7\}, \{3\}, \{6, 8\}\}$. Then some entries of $X(R, P)$ equal to 1 include $X(R, P)_{(9,9,2,0)}^{(0,7,9,7)}$ and $X(R, P)_{(1,1,8,0)}^{(0,10,1,10)}$, whereas the entries $X(R, P)_{(9,9,0,1)}^{(0,7,9,7)}$, $X(R, P)_{(9,9,9,0)}^{(0,7,9,7)}$, and

$X(R,P)^{(0,7,8,7)}_{(9,9,2,0)}$ all equal 0. Furthermore,

$$X(R,P)(v_9 \otimes v_9 \otimes v_2 \otimes v_0) = \sum_{\substack{i=1\\ i\neq 9}}^{10} v_0 \otimes v_i \otimes v_9 \otimes v_i,$$

whereas

$$X(R,P)(v_9 \otimes v_9 \otimes v_9 \otimes v_0) = 0.$$

Note that our definition of $X(R,P)$ implies that if an endomorphism X is of the form $X(R,P)$ and there are I, J such that $X^J_I = 1$, then R and P are uniquely determined by I and J. For example, let us again consider the case where $n = 10$ and $k = 4$. Suppose that $X(R,P)^J_I = 1$, where $I = (3,0,5,5)$ and $J = (0,3,2,3)$; then R must equal $\{2,5\}$ and P must be the set partition $\{\{1,6,8\},\{3,4\},\{7\}\}$. We also make the following observations about $X(R,P)$.

Proposition 8. *Suppose $X(R,P)$ is the endomorphism of $U^{\otimes k}$ defined above. Then*

(1) *$X(R,P)$ is the zero matrix iff P has more than n parts.*
(2) *$X(R,P) \in End_{S_n}(U^{\otimes k})$.*

Proof. (1) Let P be the collection of sets $P_1, P_2, \ldots, P_l$, where we assume $l \leqslant n$, and let R be the elements of Π not contained in a part of P. For $i \in \{1,2,\ldots,2k\}$, define

$$b_i = \begin{cases} 0 & \text{if } i \in R,\\ j & \text{if } i \in P_j. \end{cases}$$

Since $b_i \in \{0,1,2,\ldots,l\} \subseteq \{0,1,2,\ldots,n\}$ for all i, if $I = (b_1,\ldots,b_k)$ and $J = (b_{k+1},\ldots,b_{2k})$, $X(R,P)^J_I$ is an entry of $X(R,P)$. But we have defined the b_i in such a way that $X(R,P)^J_I = 1$, so $X(R,P)$ is not the zero matrix.

On the other hand, if P has more than n parts, there is no way for the condition

$$a_i = a_j, a_i \neq 0 \iff i, j \text{ are in the same part of } P$$

to be satisfied, since there are more parts of P than there are choices for a_i, because every nonzero a_i lies in $\{1,\ldots,n\}$. Thus, $X(R,P)^J_I = 0$ for all I and J.

(2) Let $I = (a_1,\ldots,a_k)$ and $J = (a_{k+1},\ldots,a_{2k})$, where $a_i \in \{0,1,2,\ldots,n\}$ for all i. For $\sigma \in S_n$, we define $b_i = \sigma(a_i)$, where, as before, σ fixes 0 and naturally permutes $1,2,\ldots,n$. Then $b_i = 0$ iff $a_i = 0$ and $b_i = b_j$ iff $a_i = a_j$, so $X(R,P)^{\sigma(J)}_{\sigma(I)} = X(R,P)^J_I$ for all $\sigma \in S_n$ and all k-tuples I, J. By Theorem 4, $X(R,P) \in End_{S_n}(U^{\otimes k})$ as desired. □

In fact, these particular endomorphisms $X(R,P)$ actually generate $End_{S_n}(U^{\otimes k})$.

Theorem 9. *Given an element* $(R, P) \in \Lambda(\Pi)$, *suppose* $X(R, P)$ *is the matrix defined in Definition* 6. *Then the set*

$$B = \{X(R, P) | (R, P) \in \Lambda(\Pi),\ P \textit{ has no more than } n \textit{ parts}\}$$

is a vector space basis for $End_{S_n}(U^{\otimes k})$.

Proof. First, note that by Proposition 8, part (1), no element of B is the zero matrix. Recall that for any pair of k-tuples (I, J) there is exactly one pair $(R, P) \in \Lambda(\Pi)$ such that $X(R, P)_I^J = 1$. Therefore, if $(R, P) \neq (R', P')$, the matrices $X(R, P)$ and $X(R', P')$ do not both contain an entry of 1 in the same position (I, J), and so B must be a linearly independent set.

Now let X be a matrix in $End_{S_n}(U^{\otimes k})$. In order to show that X is a linear combination of elements from B, we induct on the number of nonzero entries in X. Suppose that X has the nonzero entry $X_I^J = c_{I,J} \neq 0$. Then fix R to be the set $\{l | 1 \leqslant l \leqslant 2k, a_l = 0\}$, where, as usual, $I = (a_1, \ldots, a_k)$ and $J = (a_{k+1}, \ldots, a_{2k})$. Let P be the set partition of $\Pi \setminus R$ uniquely determined by I and J, and now consider the matrix $Y = X - c_{I,J} X(R, P)$. Since X and $X(R, P)$ are both in $End_{S_n}(U^{\otimes k})$, Y must be in the centralizer as well. Note that by Theorem 4 and our choice of $X(R, P)$,

$$X_{\sigma(I)}^{\sigma(J)} = X_I^J = c_{I,J} = c_{I,J} X(R, P)_I^J = c_{I,J} X(R, P)_{\sigma(I)}^{\sigma(J)}.$$

As a result, we have defined Y in such a way that it is an element of $End_{S_n}(U^{\otimes k})$ with fewer nonzero entries than X, since

$$Y_{I'}^{J'} = \begin{cases} 0 & \text{if } I' = \sigma(I) \text{ and } J' = \sigma(J) \text{ for some } \sigma \in S_n, \\ X_{I'}^{J'} & \text{otherwise.} \end{cases}$$

Hence, by applying our induction hypothesis to Y, we can express Y as a linear combination of elements in B; as $X = Y + c_{I,J} X(R, P)$, we are done. □

We now compute the dimension of $End_{S_n}(U^{\otimes k})$ in terms of n and k. For each subset R of Π of size i, there are exactly $\sum_{j=1}^{n} S(2k - i, j)$ set partitions of $\Pi \setminus R$ into no more than n parts, where $S(2k - i, j)$ denotes the Stirling number of the second kind. Each such set partition P indexes an endomorphism in the basis, and so we obtain the following result.

Theorem 10. *The dimension of* $End_{S_n}(U^{\otimes k})$ *is* $\sum_{i=0}^{2k} \sum_{j=1}^{n} \binom{2k}{i} S(2k - i, j)$.

As is often the situation in combinatorics, counting in a different way can yield a more tractable formula. Consider the set $\widehat{\Pi} = \{1, 2, \ldots, 2k, 2k + 1\}$. There is a natural bijection between the set partitions of $\widehat{\Pi}$ with no more than $n+1$ parts and the set partitions of subsets of Π with no more than n parts obtained as follows. Given any set partition of $\widehat{\Pi}$, one of its parts π_j contains the element $2k + 1$; let P be the collection of parts that remain in our given set partition of $\widehat{\Pi}$ once π_j has been removed. Then P is necessarily a set partition of a subset of Π with no more than n parts. (In this case, our associated set R would be the set $\pi_j \setminus \{2k+1\}$.) This mapping is easily inverted, and is hence a bijection. Thus, we have that the dimension of $End_{S_n}(U^{\otimes k})$ is also given by $\sum_{j=1}^{n+1} S(2k+1, j)$. In the case that $2k+1 \leqslant n+1$, our dimension formula simplifies further to $B(2k+1)$, the Bell number representing the total number of set partitions of $2k+1$ elements.

While the basis B given in Theorem 9 has its advantages, it is not ideal. We would like for certain canonical transformations in $End_{S_n}(U^{\otimes k})$, such as a permutation of the slots in the k-tensor, to correspond to a single basis element. For example, given what we know about other related centralizer algebras, the identity transformation *should* be represented by $X(\emptyset, \{1, k+1\}/\{2, k+2\}/\ldots/\{k, 2k\})$, but in fact this element in $End_{S_n}(U^{\otimes k})$ does not act as the identity: it annihilates many elements of $U^{\otimes k}$, such as $v_1 \otimes v_1 \otimes \cdots \otimes v_1$, for example. We therefore introduce a new basis of $End_{S_n}(U^{\otimes k})$ that, though more complicated to describe, will be a superset of the standard bases for the centralizer algebras living inside $End_{S_n}(U^{\otimes k})$, such as $\mathbb{C}S_k$, $\mathbb{C}R_k$, $B_k(n+1)$, and $P_k(n+1)$. To do this, we first define a partial ordering on $\Lambda(\Pi)$.

Definition 11. Suppose $(R', P'), (R, P) \in \Lambda(\Pi)$. We say that (R', P') is coarser than (R, P) if

(1) $R \subseteq R'$, and
(2) each nonempty part P_i' of P' is a union of parts of P.

We write $(R', P') \succeq (R, P)$.

With this definition, (R, P) is trivially coarser than itself. As an example, note that $(\{3,4,5,8\}, \{1\}/\{2,6,7\})$ is coarser than $(\{5\}, \{1\}/\{2,6\}/\{3,4,8\}/\{7\})$. We are now ready to define elements in what will be our "good" basis of $End_{S_n}(U^{\otimes k})$.

Definition 12. Let $\widehat{X}(R, P) = \sum_{\substack{(R',P') \in \Lambda(\Pi) \\ (R',P') \succeq (R,P)}} X(R', P')$.

Let us look at an example of this newly defined endormorphism; one can compare this to what we computed in Example 7.

Example 13. Let $n = 10$ and $k = 4$. Suppose $R = \{4, 5\}$ and P is the set partition $\{\{1,2,7\}, \{3\}, \{6,8\}\}$. Then some entries of $\widehat{X}(R, P)$ equal to 1 include $\widehat{X}(R, P)^{(0,7,9,7)}_{(9,9,2,0)}$, $\widehat{X}(R, P)^{(0,7,9,7)}_{(9,9,9,0)}$, $\widehat{X}(R, P)^{(0,0,9,0)}_{(9,9,0,0)}$ and $\widehat{X}(R, P)^{(0,10,1,10)}_{(1,1,8,0)}$, whereas the entries $\widehat{X}(R, P)^{(0,7,9,7)}_{(9,9,0,1)}$ and $\widehat{X}(R, P)^{(0,7,8,7)}_{(9,9,2,0)}$ all equal 0. Furthermore,

$$\widehat{X}(R, P)(v_9 \otimes v_9 \otimes v_2 \otimes v_0) = \widehat{X}(R, P)(v_9 \otimes v_9 \otimes v_9 \otimes v_0) = \sum_{i=0}^{10} v_0 \otimes v_i \otimes v_9 \otimes v_i,$$

whereas

$$\widehat{X}(R, P)(v_9 \otimes v_9 \otimes v_0 \otimes v_1) = 0.$$

Theorem 14. *The set* $\widehat{B} = \{\widehat{X}(R, P) | (R, P) \in \Lambda(\Pi)$, P *has no more than* n *parts*$\}$ *is a basis for* $End_{S_n}(U^{\otimes k})$.

Proof. A standard result in combinatorics [2] states that one can extend any partial ordering on a finite set to a linear ordering that respects the original partial ordering. So, extend the partial ordering $\succeq$ on $\Lambda(\Pi)$ to a linear ordering $\geqslant$ so that if $(R', P') \succeq (R, P)$ then $(R', P') \geqslant (R, P)$ as well. Under this linear ordering, it is clear that the matrix which expresses the elements of $\widehat{B}$ in terms of elements of B is upper-triangular with 1's on the diagonal and is therefore invertible. Since B is a basis for $End_{S_n}(U^{\otimes k})$, $\widehat{B}$ must be as well, which is what we wanted to show. □

2.3. A related diagram algebra

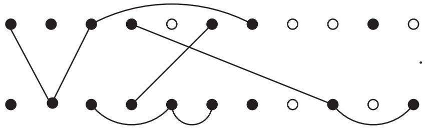
We now introduce a diagram algebra that will serve as an analog to the diagram algebras referred to in the introduction. That is, our centralizer algebra $End_{S_n}(U^{\otimes k})$ will essentially be isomorphic to a quotient of our diagram algebra. As before, let $\Pi = \{1, 2, \ldots, 2k\}$, and $\Lambda(\Pi) = \{(R, P) | R \subseteq \Pi, P$ is a set partition of $\Pi \setminus R\}$. For every element of $\Lambda(\Pi)$, we define a corresponding diagram.

Definition 15. Let $(R, P) \in \Lambda(\Pi)$. The associated diagram $d_{(R,P)} \in RP_k(x)$ is a graph on $2k$ vertices such that:

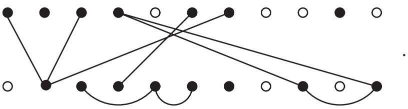
- there are two rows, each of which contains k vertices;
- the top row of vertices, from left to right, correspond to the elements $\{1, 2, \ldots, k\}$ of Π;
- the bottom row of vertices, from left to right, correspond the elements $\{k + 1, k + 2, \ldots, 2k\}$ of Π;
- there is a path between two vertices iff the corresponding elements of Π are contained in the same part of P;
- vertices corresponding to elements of R are colored white, and vertices corresponding to elements of $\Pi \setminus R$ are colored black.

We call the vertices in a diagram that are colored white *rook dots*.

Example 16. If $R = \{5, 8, 9, 11, 12, 19, 21\}$ and $P = \{1, 3, 7, 13\}/\{2\}/\{4, 20, 22\}/\{6, 15\}/\{10\}/\{14, 16, 17\}/\{18\}$, then

$d_{(R,P)} =$  .

Note that more than one graphical representation can exist for a given diagram; we call such diagrams *equivalent*. Two diagrams are equivalent exactly when their corresponding vertices are colored similarly and their connected components are identical. For example, the diagram in Example 16 is equivalent to



Definition 17. $RP_k(n+1) = span_{\mathbb{C}}\{d_{(R,P)} \mid (R, P) \in \Lambda(\Pi)\}$.

The diagram $d_{(R,P)}$ is associated with the transformation $\widehat{X}(R, P)$ in $End_{S_n}(U^{\otimes k})$. One can now use the diagram to "visualize" how the transformation $\widehat{X}(R, P)$ acts on a basis vector in $U^{\otimes k}$. The top row of the diagram corresponds to the input, and the bottom row relates to the output of the transformation. Vertices in the same component of the diagram must all correspond to the same vector, and furthermore, rook dots must correspond to the vector v_0. For instance, if we use the diagram $d_{(R,P)}$ from Example 16, then

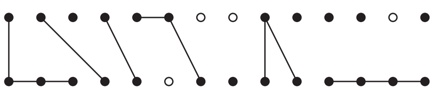
$$\begin{aligned} &d_{(R,P)}(v_{i_1} v_{i_2} \cdots v_{i_{11}}) \\ &\quad = \delta_{i_5,0}\delta_{i_8,0}\delta_{i_9,0}\delta_{i_{11},0}\delta_{i_1,i_3}\delta_{i_3,i_7} \sum_{l=0}^{n} \sum_{m=0}^{n} v_0 v_{i_1} v_l v_{i_6} v_l v_l v_m v_0 v_{i_4} v_0 v_{i_4}, \end{aligned}$$

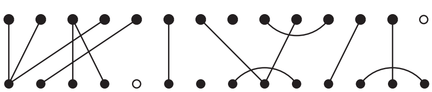
where the tensor signs between the vectors have been omitted for compactness.

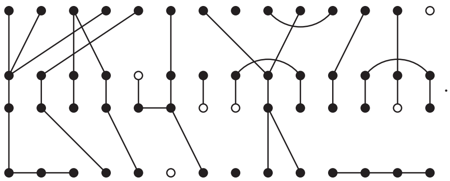
The following combinatorial description of the product of two diagrams is exactly what is needed to match the product of the corresponding centralizer algebra transformations the diagrams correspond to. To compute the product $d_1 d_2$ of two diagrams in the basis of $RP_k(n+1)$:

- Place d_2 on top of d_1, and then connect each vertex in the bottom row of d_2 with the corresponding one in the top row of d_1.
- Let γ be the number of connected components in the concatenation that
  - do not contain a rook dot, and
  - do not contain vertices in the top or bottom row (that is, the component must be contained entirely in the middle two rows of the concatenation).
- Let d_3 be the diagram obtained by using only the top and bottom row of the concatenation, with the imposed connections. Note: any vertex in the top or bottom row of the concatenation that is connected to a rook dot becomes a rook dot in d_3.

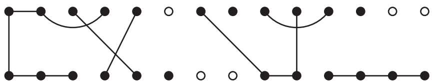
Then, the product $d_1 d_2$ is $(n+1)^{\gamma} d_3$. For example, if

$d_1 =$  and

$d_2 =$  , then we concatenate to get the diagram



From this concatenation, we conclude that the product $d_1 d_2$ equals

$(n+1)$ 

The combinatorial argument given after Theorem 10 can be used to show that, as a vector space, the dimension of $RP_k(n+1)$ is the Bell number $B(2k+1)$. When $2k+1 \leqslant n+1$, the Bell number $B(2k+1)$ is also the dimension of $End_{S_n}(U^{\otimes k})$, and thus:

Theorem 18 (*Schur–Weyl duality for the rook partition algebras*). *S_n and $RP_k(n+1)$ generate full centralizers of each other on $U^{\otimes k}$. In particular, when $n \geqslant 2k$, $End_{S_n}(U^{\otimes k}) \cong RP_k(n+1)$.*

By replacing the parameter $n+1$ with the parameter x, where $x \in \mathbb{C}$, we get the general rook partition algebras. Observe that $d_{(\emptyset,P)}$ is an element of $P_k(x)$ and that in fact $\{d_{(\emptyset,P)} | (\emptyset, P) \in \Lambda(\Pi)\}$ is the standard basis for $P_k(x)$. Moreover, contained in the set of standard basis diagrams for $P_k(x)$ is

- the standard basis for $B_k(x)$ (namely, those diagrams where the $2k$ vertices are paired up into k components of 2 vertices each), and
- the standard diagram basis for $\mathbb{C}S_k$ (those diagrams where the $2k$ vertices are paired up into k components, where each component contains exactly one vertex from the top row and one vertex from the bottom row).

$RP_k(x)$ also contains the standard basis for $RB_k(x)$ (diagrams in which each component is either a singleton rook dot or consists of two vertices) and the standard diagram basis for $\mathbb{C}R_k$ (diagrams in which each component is either a singleton rook dot or consists of two vertices, one from each row of the diagram).[2] Furthermore, the algorithm to multiply diagrams in $RP_k(x)$ restricts appropriately, so that if d_1, d_2 are diagrams in the diagram algebra Y, where Y is one of $\mathbb{C}S_k$, $B_k(x)$, $P_k(x)$, $\mathbb{C}R_k$, or $RB_k(x)$, computing $d_1 d_2$ in $RP_k(x)$ is the same as computing $d_1 d_2$ using multiplication algorithms in the literature for Y. In this way, we have the inclusion of algebras

$$\begin{array}{ccccc} \mathbb{C}R_k & \subset & RB_k(x) & \subset & RP_k(x) \\ \cup & & \cup & & \cup \\ \mathbb{C}S_k & \subset & B_k(x) & \subset & P_k(x) \end{array}.$$

[2] In the literature, these rook dots are not colored white; they need not be, since isolated vertices only come in one "flavor" in those cases.

3. Representations of the rook partition algebra

Schur–Weyl duality is a powerful tool because it allows us to study the representations of one algebraic object by using knowledge of the representation theory of its centralizer. For example, since $U^{\otimes k}$ is a completely reducible S_n-module, $End_{S_n}(U^{\otimes k})$ is a semisimple algebra. So when $n \geqslant 2k$, $RP_k(n+1)$ is semisimple. In fact, Martin and Saleur have shown that

Theorem 19 (*Martin and Saleur [17]*). *For each integer $k \geqslant 0$, $RP_k(x)$ is semisimple over $\mathbb{C}(x)$, and $RP_k(\xi)$ is semisimple over $\mathbb{C}$ whenever ξ is not an integer in the range* $[0, 2k-1]$.

Additionally Halverson and Ram give conditions in [8] for testing semisimplicity in the case ξ is an integer in the range $[0, 2k-1]$. For the remainder of this paper, we will assume that we are in the case $n \geqslant 2k$, so that $End_{S_n}(U^{\otimes k}) \cong RP_k(n+1)$.

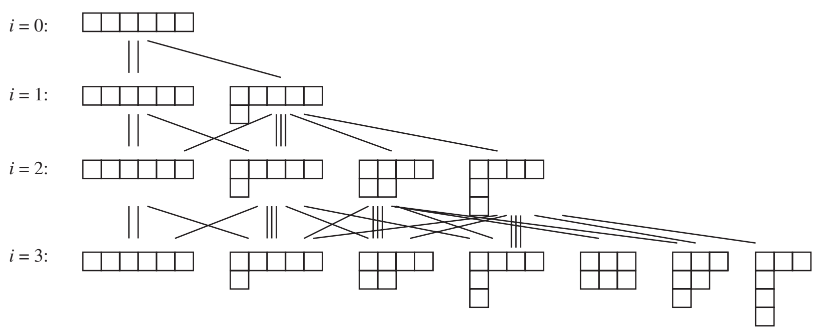
The irreducible representations of S_n are indexed by the partitions $\lambda \vdash n$, and we write S^λ to denote the irreducible S_n-module indexed by λ. As an S_n-module,

$$U \cong V \oplus \mathbb{C} \cong (S^{(n-1,1)} \oplus S^{(n)}) \oplus S^{(n)}.$$

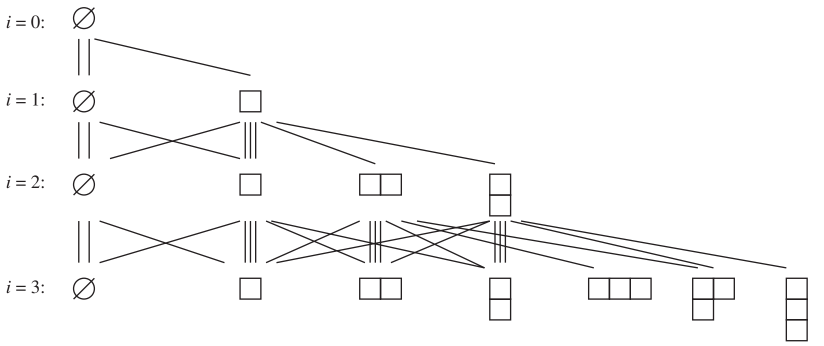
We now consider the *Bratelli diagram*, a diagram in which the ith level lists the irreducible S_n-modules that appear in $U^{\otimes i}$, and each edge from a shape λ in the ith level to a shape μ in the $(i+1)$th level represents a factor of S^μ in the decomposition of $S^\lambda \otimes U$. As a result, the number of paths from the 0th level of the diagram to a particular shape λ at the ith level gives the multiplicity of S^λ in the decomposition of $U^{\otimes i}$. Using the rules for decomposing tensor products of irreducible S_n-modules, we note that

$$S^\lambda \otimes U \cong S^\lambda \oplus \sum_{\mu=(\lambda^-)^+} S^\mu,$$

where the sum is over all partitions of n that are obtained by first removing a box from λ to get λ^- and then adding a box to λ^- to get μ. The first few levels of the Bratelli diagram for S_6 are shown below.



Note that, when $n \geqslant 2i$, the ith level of the Bratelli diagram contains all partitions $\mu \vdash n$ such that $|\mu^*| \leqslant i$, where μ^* is just the partition μ with its first part removed. This observation allows us to simplify our Bratelli diagrams in the case $n \geqslant 2i$ by substituting μ^* for μ; for instance, the Bratelli diagram above can be represented as follows.



Double centralizer theory tells us that

- The irreducible representations of $RP_k(n+1)$ are indexed by the same set that indexes the irreducible S_n-modules of $U^{\otimes k}$. So the kth level of the Bratelli diagram gives a complete set of irreducible $RP_k(n+1)$-modules, and this set is can be indexed either by $\{\lambda | 0 \leqslant |\lambda| \leqslant k\}$ or $\{\mu \vdash n | 0 \leqslant |\mu^*| \leqslant k\}$.
- The multiplicity of S^λ in $U^{\otimes k}$ (which equals the number of paths from the top of the Bratelli diagram to the shape λ on the kth level) is the dimension of the irreducible $RP_k(n+1)$-module indexed by λ. For example, using the Bratelli diagram above, we see that the dimensions of the irreducible $RP_2(7)$-modules are, reading from left to right, 5, 5, 1, and 1, and $5^2+5^2+1^2+1^2=52=B(5)$; the dimensions of the irreducible $RP_3(7)$-modules are, from left to right, 15, 22, 9, 9, 1, 2, 1 and $15^2+22^2+9^2+9^2+1^2+2^2+1^2=877=B(7)$.
- The edges in the Bratelli diagram from the ith level to the $(i-1)$th level give the branching rules from $RP_i(n+1)$ to $RP_{i-1}(n+1)$. (One can embed $RP_{i-1}(n+1)$ into $RP_i(n+1)$ simply by adding an ith column with a single vertical edge to a diagram in $RP_{i-1}(n+1)$.) For example, the irreducible $RP_3(7)$-module indexed by $\lambda=(1,1)$ (or, $\mu=(4,1,1)$) decomposes as an $RP_2(7)$-module into 3 copies of the irreducible $RP_2(7)$-module indexed by $\lambda=(1,1)$ (or, $\mu=(4,1,1)$), one copy of the irreducible $RP_2(7)$-module indexed by (2) (or, (4, 2)), and one copy of the irreducible $RP_2(7)$-module indexed by (1) (or, (5, 1)).

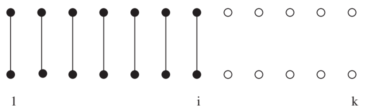
Martin and Saleur [17] have shown that the irreducible $RP_k(x)$-modules are also indexed by $\{\lambda | 0 \leqslant |\lambda| \leqslant k\}$ and that the dimensions and branching rules are the same as those for $RP_k(n+1)$. We proceed by giving a construction of the irreducible $RP_k(x)$-modules that is a bit cleaner than the one found in [17].

Definition 20. Let d be a diagram in $RP_k(x)$. A through class of d is a connected component of d containing vertices in both the top and bottom row of d. Let $tc(d)$ denote the number of through classes of d.

Note that multiplying diagrams can never increase the number of through classes; i.e.,

$$tc(dd') \leqslant \min\{tc(d), tc(d')\}.$$

Let C_k be shorthand notation for $RP_k(x)$. We now define the diagrams

$A_i =$  $, \quad 0 \le i \le k,$

and consider $C_k A_i C_k$, the ideal of C_k consisting of diagrams with no more than i through classes. By letting i vary, we have the following chain of ideals:

$$C_k A_0 C_k \lhd C_k A_1 C_k \lhd \cdots \lhd C_k A_{k-1} C_k \lhd C_k A_k C_k = C_k,$$

and hence a collection of quotient modules

$$M_i = C_k A_i C_k / C_k A_{i-1} C_k.$$

The module M_i can be thought of as the set of diagrams in C_k with *exactly* i through classes: all diagrams in $C_k A_i C_k$ contain no more than i through classes and those diagrams with fewer through classes are absorbed into the quotient.

If d_1 is a diagram in $RP_{k_1}(x)$ and d_2 is a diagram in $RP_{k_2}(x)$, we can define $d_1 \otimes d_2$ as the diagram in $RP_{(k_1+k_2)}(x)$ obtained by placing d_1 to the left of d_2. Let $d^{\otimes h}$ be shorthand notation for the diagram corresponding to tensoring d with itself h times. We can embed the symmetric group S_i into C_k using the following map:

$$\sigma \hookrightarrow \sigma \otimes F^{\otimes k-i},$$

where

$F =$  .

Since the element $\sigma \otimes F^{\otimes k-i}$ has exactly i through classes, it is essentially a nontrivial element of M_i. Thus, we have a copy of the group algebra $\mathbb{C}[S_i]$ living inside M_i. We will exploit our knowledge of the representation theory of $\mathbb{C}[S_i]$ to construct minimal idempotents of M_i. Suppose that e_λ is a minimal idempotent of $\mathbb{C}[S_i]$. We will abuse notation somewhat, by letting $\widehat{e_\lambda} \in M_i$ be the element corresponding to $e_\lambda \otimes F^{\otimes k-i} \in C_k$. A straightforward computation shows that $\widehat{e_\lambda}$ and A_i are idempotents and

$$A_i \widehat{e_\lambda} = \widehat{e_\lambda} A_i = \widehat{e_\lambda}.$$

In fact:

Theorem 21. *Let $\lambda \vdash i$. Then $\widehat{e_\lambda}$ is a minimal idempotent of M_i.*

Proof. Suppose $\widehat{e_\lambda}$ is not minimal; that is, assume that

$$\widehat{e_\lambda} = u_1 + u_2,$$

where u_1 and u_2 are orthogonal idempotents in M_i. The idempotent u_1 has the form $\sum_{d \in M_i} \alpha_d d$, where the α_d are constants depending on d. We then have the following string of equalities:

$$\begin{aligned} u_1 &= (u_1 + u_2)u_1(u_1 + u_2) \\ &= \widehat{e}_\lambda u_1 \widehat{e}_\lambda \\ &= A_i \widehat{e}_\lambda u_1 \widehat{e}_\lambda A_i \\ &= A_i \left(\sum_{d \in M_i} \widehat{e}_\lambda (\alpha_d d) \widehat{e}_\lambda \right) A_i \\ &= \sum_{d' \in M_i} \beta_{d'} A_i d' A_i \end{aligned}$$

for some constants $\beta_{d'}$. Since the product $A_i d' A_i$ necessarily has rook dots in the last $k-i$ columns, every diagram d in the sum $u_1 = \sum_{d \in M_i} \alpha_d d$ has rook dots in its last $k-i$ columns. Let d be an arbitrary diagram in the sum $\sum_{d \in M_i} \alpha_d d$ with $\alpha_d \neq 0$. Since d contains exactly i through classes, this implies that the first i columns of d must correspond to a permutation in S_i. In other words,

$$u_1 = \sum_{\sigma \in S_i} \alpha_\sigma \sigma \otimes F^{\otimes k-i}.$$

Similarly, we can conclude that

$$u_2 = \sum_{\rho \in S_i} \beta_\rho \rho \otimes F^{\otimes k-i}.$$

Now let $e_1 = \sum_{\sigma \in S_i} \alpha_\sigma \sigma$ and $e_2 = \sum_{\rho \in S_i} \beta_\rho \rho$ be elements of $\mathbb{C}[S_i]$. Since u_1 and u_2 are orthogonal idempotents in M_i, e_1 and e_2 must be orthogonal idempotents in $\mathbb{C}[S_i]$. However, $e_\lambda = e_1 + e_2$, contradicting the fact that e_λ is a minimal idempotent in $\mathbb{C}[S_i]$. $\square$

Since $\widehat{e}_\lambda$ is a minimal idempotent of M_i, we have that $M_i \widehat{e}_\lambda$ is an irreducible M_i-module and that

Corollary 22. *If $\lambda \vdash i$, where $0 \leqslant i \leqslant k$, then $M^\lambda = M_i \widehat{e}_\lambda$ is an irreducible $RP_k(x)$-module.*

We now have a collection of irreducible $RP_k(x)$ modules indexed by $\{\lambda \vdash i \mid 0 \leqslant i \leqslant k\}$; if these modules are pairwise non-isomorphic, we will have constructed all the irreducible $RP_k(x)$-modules.

Proposition 23. *Let $\lambda \vdash i$ and $\mu \vdash j$, where $0 \leqslant i, j \leqslant k$. Then $M^\lambda \ncong M^\mu$ as $RP_k(x)$-modules if $\lambda \neq \mu$.*

Proof. Let us first deal with the case that $i \neq j$. Without loss of generality, suppose that $i < j$. Then the element A_i annihilates $M^\mu = M_j \widehat{e}_\mu$, because $tc(A_i d) \leqslant i < j$, but does not annihilate $M_i \widehat{e}_\lambda = M^\lambda$.

Now suppose that $i = j$ and assume, towards a contradiction, that $M^\lambda \cong M^\mu$ as $RP_k(x)$-modules. Consider the copy of the group algebra $\mathbb{C}[S_i]$ inside M_i that is generated by $\{\sigma \otimes F^{\otimes k-i} \mid \sigma \in S_i\}$ and denote it $\mathbb{S}_i$. The modules M^λ and M^μ are then also isomorphic as $\mathbb{S}_i$-modules; call this isomorphism ϕ. Since $A_i \in \mathbb{S}_i$, we have that

$$\phi(A_i M^\lambda) = A_i \phi(M^\lambda) = A_i M^\mu. \tag{1}$$

Note, however, that $A_i M_i A_i = \mathbb{S}_i$: any diagram of the form $A_i d A_i$ has rook dots in the last $k-i$ columns, and so if $A_i d A_i$ has exactly i through classes, then it is of the form $\sigma \otimes F^{\otimes k-i}$ for some $\sigma \in S_i$. So $A_i M^\lambda = A_i M_i \widehat{e}_\lambda = A_i M_i A_i \widehat{e}_\lambda = \mathbb{S}_i \widehat{e}_\lambda \cong S^\lambda$. Similarly, $A_i M^\mu \cong S^\mu$. Substituting into (1), we get that $S^\lambda \cong S^\mu$, which holds only if $\lambda = \mu$, a contradiction. $\square$

4. Characters of the rook partition algebra

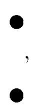
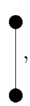
Our method of computing character values follows [7] closely. There are three basic steps to this process. First, find a "nice" set of elements on which it is sufficient to compute the character values. These elements play the same role in the algebra $RP_k(x)$ as conjugacy class representatives do in the representation theory of finite groups. Second, view $U^{\otimes k}$ as a $RP_k(n+1) \times S_n$-bimodule, and compute the bitrace of this bimodule in two different ways. This gives us a Frobenius-type formula for the characters of $RP_k(n+1)$. Finally, we can use this formula to obtain a combinatorial formula for computing character values: a Murnaghan–Nakayama rule for the rook partition algebras.

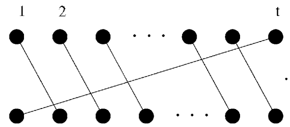
4.1. Conjugacy class analogs

Since the character is constant on conjugacy classes, when computing characters of group representations it is sufficient to compute the character value for just one element of each conjugacy class. Since not all elements in our algebra are invertible, we do not have conjugacy classes per se, so our goal is to find a set of "conjugacy class analogs". That is, we would like to find a collection of elements $\mathcal{C}$ in $RP_k(x)$ such that, given any diagram d in $RP_k(x)$, $\chi(d) = r_d \chi(c_d)$, where χ is any character of $RP_k(x)$, c_d is an element in $\mathcal{C}$ that can be determined from d and r_d is a scalar that can be determined from d.

4.1.1. Standard elements

Since we will refer to the following diagrams often, we give them labels. Let

$E =$ , $\gamma_1 =$ , and for $t > 1$ let

$\gamma_t =$  .

Recall from the previous section that if d_1 is a diagram in $RP_{k_1}(x)$ and d_2 is a diagram in $RP_{k_2}(x)$, then $d_1 \otimes d_2$ is the diagram in $RP_{(k_1+k_2)}(x)$ obtained by placing d_1 to the left of d_2, and $d^{\otimes h}$ is shorthand notation for the diagram corresponding to tensoring d with itself h times.

Then, for partitions $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ of m, we can define

$$\gamma_\mu = \gamma_{\mu_1} \otimes \gamma_{\mu_2} \otimes \cdots \otimes \gamma_{\mu_l},$$

and when $m \leqslant k$, we define the following diagram $d_\mu \in RP_k(x)$:

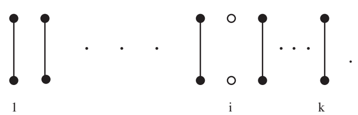
$$d_\mu = \gamma_\mu \otimes E^{\otimes k-m}.$$

We will show that these diagrams $\{d_\mu | \mu \vdash m, 0 \leqslant m \leqslant k\}$ form our collection $\mathcal{C}$ of conjugacy class analogs. Note that $\mathcal{C}$ has the same index set as the set of irreducible representations of $RP_k(x)$, so our character table will be square.

4.1.2. Blocks

Since all the elements in $\mathcal{C}$ are actually partition diagrams-that is, they all live in $P_k(x)$, our first step in computing $\chi(d)$ for an arbitrary diagram $d \in RP_k(x)$ is to associate d with a diagram in $P_k(x)$. We accomplish this goal in stages, gradually eliminating the rook dots in d. Let us start by defining a few basic diagrams that contain rook dots. Let

$F =$  , $L =$  , $U =$  and

$F_i =$  .

We are now ready for our first lemma.

Lemma 24. *Let $d \in RP_k(x)$. Suppose that $d = \tilde{d} \otimes F^{\otimes k-r}$, where $\tilde{d}$ is a partition diagram in $P_r(x)$ for some $r < k$. Then $\chi(d) = \frac{1}{x^{k-r}} \chi(\tilde{d} \otimes E^{\otimes k-r})$.*

Proof. Let $a = (\gamma_1)^{\otimes r} \otimes L^{\otimes k-r}$ and let $b = (\gamma_1)^{\otimes r} \otimes U^{\otimes k-r}$; note that both a and b act as the identity in the first r components. Then, we have that

$$adb = \tilde{d} \otimes E^{\otimes k-r}$$

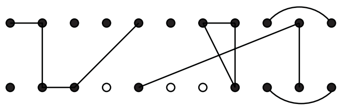
and

$$dba = x^{k-r} d.$$

Since $\chi(a(db)) = \chi((db)a) = x^{k-r}\chi(d)$, the result follows. □

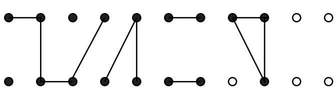
Our next goal is to show that if d is a diagram in $RP_k(x)$, then there is a diagram $\tilde{d}$ in $P_r(x)$, for some $r \leqslant k$, such that $\chi(d) = \chi(\tilde{d} \otimes F^{\otimes k-r})$. Then, we can use Lemma 24 to associate d with a diagram in $\mathcal{C}$. In order to find $\tilde{d}$, we introduce the notion of the *blocks* of a diagram d. Connect the vertices of each column of d with dotted lines. The connected components of this new graph are defined to be the *blocks* of d. (While we connect vertices in the same column with edges to determine blocks, when we refer to the blocks of a diagram, we do not include those added edges.)

Example 25. The diagram

$d =$  ,

has 5 blocks: columns 1, 2, 3, 5, and 10; column 4; column 6; columns 7 and 8; and columns 9 and 11.

We can obtain a new diagram with better-ordered columns simply by conjugating by an appropriate element π of S_k, since $\pi d \pi^{-1}$ simply rearranges the columns of d according to the permutation π. Since conjugation does not affect the character value—that is, $\chi(\pi d \pi^{-1}) = \chi(d)$-we may work only with diagrams such that all the columns in a given block appear in consecutive order, and furthermore all the columns in a block with rook dots are at the right-most side of the diagram. For instance, the diagram in Example 25 has the same character as the conjugate diagram

$\tilde{d} =$  . (2)

Proposition 26. *Let $d \in RP_k(x)$, and suppose $d = d' \otimes b$, where b is a block that has s columns and at least one rook dot. Then $\chi(d) = \chi(d' \otimes F^{\otimes s})$.*

Proof. If b contains only rook dots, we are trivially done. (In this case, $b = F$ because any column containing two rook dots is a block unto itself.) If not, there is a column of b (and hence, of d) that has a rook dot in one row but not in the other. If the rook dot is in the top row of the ith column of d, then $dF_i = d$, while $F_i d = d' \otimes F^{\otimes s}$; if the rook dot is in the bottom row of the ith column of d, then $F_i d = d$, whereas $dF_i = d' \otimes F^{\otimes s}$. Since $\chi(dF_i) = \chi(F_i d)$, the proposition is proven. □

We may apply this procedure, block by block, to the blocks of d that contain rook dots, so that the end result is

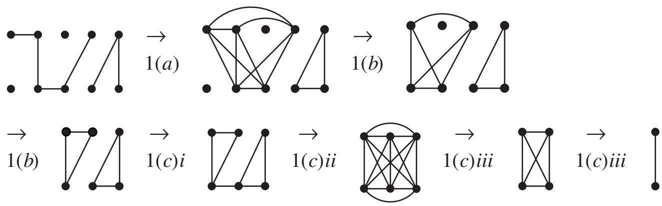
Theorem 27. *Let $d \in RP_k(x)$, and let r equal the number of columns in blocks that do not contain a rook dot. Then $\chi(d) = \chi(\tilde{d} \otimes F^{\otimes k-r})$ for some $\tilde{d} \in P_r(x)$.*

Note that we may assume that the $\tilde{d}$ in the theorem above has its columns clustered into blocks as in (2). We now consider the *block type* of a diagram in $RP_k(x)$. The block type is an assignment of a nonnegative integer to those blocks of a diagram *that do not contain a rook dot* using the following algorithm (cf. [7]):

(1) While a block is not E or conjugate to γ_t for some $t \geqslant 1$, do the following:
  (a) Add any edges to the block implied by transitivity; that is, add edges so that any two vertices in the block connected by a path are now connected by an edge.
  (b) If the block has an isolated vertex, then delete that column, also removing any edges incident to the other vertex in that column.
  (c) If the block has a horizontal edge, then
    (i) connect the corresponding vertices in the opposite row by an edge,
    (ii) add any edges that are implied by transitivity, and then
    (iii) remove either one of the columns containing the original horizontal edge.

(2) Assign a block that is E type 0.
(3) Assign a block conjugate to γ_t type t.

Applying this algorithm to the first block in (2), we see it has type 1.



The sequence of positive integer types of a diagram d yields a partition μ with nonzero parts that we call the *block type* of d. It will also be important to keep track of the number of columns one removes in the algorithm. (When $d \in P_k(x)$, this number is $k - |\mu| -$ {the number of parts of type 0}.) We now recall the following proposition from [7].

Proposition 28. *Let d be a diagram in $P_k(x)$ with block type μ such that s columns were eliminated in the block type algorithm. Suppose χ is any character of $P_k(x)$, then*

$$\chi(d) = \frac{1}{x^s}\chi(d_\mu),$$

where $d_\mu = \gamma_\mu \otimes E^{\otimes k-|\mu|}$.

The proof of this result, like the proofs of Lemma 24 and Proposition 26, capitalizes on multiplying d on the left and right by well-chosen diagrams and the fact that $\chi(ab) = \chi(ba)$. Since neither trick is specific to $P_k(x)$, Halverson's proof can be used verbatim to show an identical result for partition diagrams in $RP_k(x)$. Furthermore, since we can work with individual blocks independently, we have:

Proposition 29. *Let d be a diagram in $RP_k(x)$ with block type μ such that s columns were eliminated in the block type algorithm. Suppose and χ is any character of $RP_k(x)$, then*

$$\chi(d) = \frac{1}{x^s}\chi(d_\mu \otimes d'),$$

where $d_\mu = \gamma_\mu \otimes E^{\otimes k-|\mu|}$, and d' consists of the blocks of d that contain rook dots.

Combining all the results in this section gives us Theorem 30, the main result we wanted to show: that for any diagram d in $RP_k(x)$, $\chi(d) = r_d c_d$, where r_d is a scalar dependent on d and c_d is a standard diagram in $\mathcal{C}$.

Theorem 30. *Let d be a diagram in $RP_k(x)$ such that*

- *the number of columns in blocks that do not contain a rook dot is r,*
- *of these columns, s were eliminated in the block type algorithm,*
- *d has block type μ, where $\mu \vdash m$.*

Let χ be any character of $RP_k(x)$. Then

$$\chi(d) = \frac{1}{x^{k-r+s}}\chi(d_\mu) = \frac{1}{x^{k-r+s}}\chi(\gamma_\mu \otimes E^{\otimes k-m}).$$

Proof.

$$\begin{aligned}
\chi(d) &= \frac{1}{x^s}\chi(\gamma_\mu \otimes E^{\otimes r-m} \otimes d') \quad \text{(by Proposition 29)}\\
&= \frac{1}{x^s}\chi(\gamma_\mu \otimes E^{\otimes r-m} \otimes F^{\otimes k-r}) \quad \text{(by Theorem 27)}\\
&= \frac{1}{x^s}\,\frac{1}{x^{k-r}}\chi(\gamma_\mu \otimes E^{\otimes r-m} \otimes E^{\otimes k-r}) \quad \text{(by Lemma 24)}\\
&= \frac{1}{x^{k-r+s}}\chi(\gamma_\mu \otimes E^{\otimes k-m}).
\end{aligned}$$

Note that, since it is sufficient to compute rook partition algebra characters on partition algebra diagrams, one can in fact now compute these characters by using the following two theorems: the first expresses an irreducible $RP_k(n+1)$-module as a sum of irreducible $P_k(n+1)$-modules, and the second gives a Murnaghan–Nakayama formula for computing partition algebra characters. Nonetheless, we will continue with our program in order to obtain explicit Frobenius and Murnaghan–Nakayama formulas for the rook partition algebra. $\square$

Theorem 31 (*Martin [15]*). *Let $\lambda \vdash r$, $0 \leqslant r \leqslant k$, and suppose V^λ is the irreducible $RP_k(n+1)$-module indexed by λ. Then, as a $P_k(n+1)$-module,*

$$V^\lambda \cong W^\lambda \oplus \sum_{\substack{\mu=\lambda^+\\ |\lambda^+|\leqslant k}} W^\mu,$$

where W^{μ} is the irreducible $P_k(n+1)$-module indexed by μ, and the sum is over all partitions λ^+, obtained by adding a box to λ, of integers no larger than k.

Theorem 32 (*Halverson [7]*). [3] *Let $\lambda \vdash n$ with $|\lambda^*| \leqslant k$, where λ^* is the partition obtained from λ by removing its first part λ_1. Let μ be a composition with $0 \leqslant |\mu| \leqslant k$, and let $\bar{\mu}$ be the composition obtained from μ by removing a part of size r. Then*

$$\chi^{\lambda}_{P_k(n)}(\gamma_\mu \otimes E^{\otimes k-|\mu|}) = \sum_{d|r} \sum_{\substack{\delta=(\lambda^{-d})^{+d} \\ |\delta^*| \leqslant k-r}} (-1)^{ht(\lambda/\lambda^{-d})}(-1)^{ht(\delta/\lambda^{-d})}\chi^{\delta}_{P_{k-r}(n)}(\gamma_{\bar{\mu}} \otimes E^{\otimes k-r-|\bar{\mu}|}),$$

where the outer sum is over all divisors d of r and the inner sum is over all $\delta \vdash n$ with $|\delta^*| \leqslant k-r$ such that δ is obtained from λ by removing a border strip of size d to obtain the partition λ^{-d} and then adding a border strip of size d to λ^{-d} to obtain $\delta = (\lambda^{-d})^{+d}$.

4.2. A Frobenius formula for the rook partition algebra

We now view $U^{\otimes k}$ as a $S_n \times RP_k(n+1)$-bimodule. We know from double centralizer theory that, when $2k+1 \leqslant n+1$,

$$U^{\otimes k} \cong \sum_{\substack{\lambda \vdash n \\ |\lambda^*| \leqslant k}} S^{\lambda} \times V^{\lambda}, \tag{3}$$

where S^{λ} is the irreducible S_n-module indexed by λ and V^{λ} is the irreducible $RP_k(n+1)$-module indexed by λ. We now take the trace of both sides of (3). On the one hand, we get that the bitrace of (σ, d) is

$$btr(\sigma, d) = \sum_{\substack{\lambda \vdash n \\ |\lambda^*| \leqslant k}} \chi^{\lambda}_{S_n}(\sigma)\chi^{\lambda}_{RP_k(n+1)}(d). \tag{4}$$

On the other hand, we can directly compute the bitrace on $U^{\otimes k}$ using the definition

$$btr(\sigma, d) = \sum_I \sigma(d \cdot v_I)|_{v_I},$$

where, as before, $I = (i_i, i_2, \ldots, i_k)$ is a k-tuple of elements from the set $\{0, 1, 2, \ldots, n\}$ and v_I denotes the k-tensor $v_{i_1} \otimes \cdots \otimes v_{i_k}$. First, a definition.

Definition 33. Let $\sigma \in S_n$ be of cycle type $\rho = (\rho_1, \rho_2, \ldots)$ and suppose r is a positive integer. We define $\tilde{f}_r(\sigma)$ to be the sum

$$\tilde{f}_r(\sigma) = 1 + \sum_{i=1}^{l(\rho)} p_r(1, \omega_{\rho_i}, \omega^2_{\rho_i}, \ldots, \omega^{\rho_i-1}_{\rho_i}),$$

[3] See Section 4.3 for definitions of notation and terminology in this theorem.

where p_r is the power symmetric function and $\omega_j = e^{\frac{2\pi i}{j}}$ is a primitive jth root of unity.

Note that $\tilde{f}_r(\sigma) - 1$ is precisely the definition of $f_r(\sigma)$ found in Theorem 3.2.2 of [7].

Theorem 34. *Let $\sigma \in S_n$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_l) \vdash m$, where $0 \leqslant m \leqslant k$. Then*

$$btr(\sigma, d_\mu) = (n+1)^{k-m} \tilde{f}_\mu(\sigma),$$

where $\tilde{f}_\mu(\sigma) = \tilde{f}_{\mu_1}(\sigma) \tilde{f}_{\mu_2}(\sigma) \cdots \tilde{f}_{\mu_l}(\sigma)$.

Proof. Suppose d_1 has n_1 columns and d_2 has $n_2 = k - n_1$ columns. Then $btr(\sigma, d_1 \otimes d_2) = btr(\sigma, d_1) btr(\sigma, d_2)$, where $\sigma \times d_1$ acts on the first n_1 slots of $U^{\otimes k}$ and $\sigma \times d_2$ acts on the last n_2 slots of $U^{\otimes k}$. It follows that

$$\begin{aligned} btr(\sigma, d_\mu) &= btr(\sigma, \gamma_\mu)(btr(\sigma, E))^{k-m} \\ &= btr(\sigma, \gamma_{\mu_1}) btr(\sigma, \gamma_{\mu_2}) \cdots btr(\sigma, \gamma_{\mu_l})(btr(\sigma, E))^{k-m}. \end{aligned}$$

To compute $btr(\sigma, E)$, we first recall that $Ev_i = \sum_{j=0}^{n} v_j$ for any $i \in \{0, 1, \ldots, n\}$ and that $\sigma v_i = v_{\sigma(i)}$, where $\sigma(0)$ is defined to be 0. Then

$$btr(\sigma, E) = \sum_{l=0}^{n} \sigma(E v_l)|_{v_l} = \sum_{l=0}^{n} \sum_{j=0}^{n} \sigma v_j|_{v_l} = \sum_{l=0}^{n} \sum_{j=0}^{n} v_{\sigma(j)}|_{v_l} = \sum_{l=0}^{n} 1 = n+1.$$

Thus, we now have that $btr(\sigma, d_\mu) = (n+1)^{k-m} btr(\sigma, \gamma_\mu)$.

It remains to show that $btr(\sigma, \gamma_c) = \tilde{f}_c(\sigma)$.

$$\begin{aligned} btr(\sigma, \gamma_c) &= \sum_{0 \leqslant i_1, i_2, \ldots, i_c \leqslant n} \sigma(\gamma_c \cdot v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_c})|_{v_{i_1} \otimes \cdots \otimes v_{i_c}} \\ &= \sum_{0 \leqslant i_1, i_2, \ldots, i_c \leqslant n} \sigma(v_{i_c} \otimes v_{i_1} \otimes \cdots \otimes v_{i_{c-1}})|_{v_{i_1} \otimes \cdots \otimes v_{i_c}} \\ &= \sum_{0 \leqslant i_1, i_2, \ldots, i_c \leqslant n} v_{\sigma(i_c)} \otimes v_{\sigma(i_1)} \otimes \cdots \otimes v_{\sigma(i_{c-1})}|_{v_{i_1} \otimes \cdots \otimes v_{i_c}} \end{aligned}$$

Note that every c-tuple $(i_1, i_2, \ldots, i_c)$ contributes either a coefficient of 1 or 0 to the sum in the above equation. Clearly, we get a nonzero contribution to the sum if and only if $\sigma(i_c) = i_1, \sigma(i_1) = i_2, \ldots,$ and $\sigma(i_{c-1}) = i_c$. What are the implications of this string of equalities? First, since $i_1 \overset{\sigma}{\mapsto} i_2 \overset{\sigma}{\mapsto} i_3 \overset{\sigma}{\mapsto} \cdots \overset{\sigma}{\mapsto} i_c \overset{\sigma}{\mapsto} i_1$, we have that $i_1, i_2, \ldots, i_c$ must either all be zero or they must all be in the same cycle of σ. So we can now focus just on the case where none of the elements in the c-tuple $(i_1, i_2, \ldots, i_c)$ are 0. If we apply σ iteratively, we see that $\sigma^c(i_j) = i_j$ for all $j = 1, 2, \ldots, c$, and thus the length of the cycle of σ that contains $i_1, \ldots, i_c$ must divide c. Let $\sigma_d = (j_1, j_2, \ldots, j_d)$ be a cycle of σ where $d|c$. Then, for every element j_l of σ_d, there is exactly one c-tensor that begins with j_l that is fixed by $\sigma\gamma_c$, namely the one with $v_{j_l} \otimes \cdots \otimes v_{j_d} \otimes v_{j_1} \otimes \cdots \otimes v_{j_{l-1}}$ repeated $\frac{c}{d}$ times.

So if the cycle type of σ is $(\rho_1, \rho_2, \ldots,)$, then $btr(\sigma, \gamma_c)$ equals the sum over the ρ_i, where ρ_i divides c, plus one (to account for the vector $v_0 \otimes \cdots \otimes v_0$).

On the other hand,

$$\begin{aligned}
\tilde{f}_c(\sigma) &= 1 + \sum_{i=1}^{l(\rho)} p_c\big(1, \omega_{\rho_i}, \omega_{\rho_i}^2, \ldots, \omega_{\rho_i}^{\rho_i - 1}\big) \\
&= 1 + \sum_{i=1}^{l(\rho)} \big(1 + \omega_{\rho_i}^c + \omega_{\rho_i}^{2c} + \cdots + \omega_{\rho_i}^{(\rho_i - 1)c}\big) \qquad (5) \\
&= 1 + \sum_{i=1}^{l(\rho)} \frac{\omega_{\rho_i}^{c(\rho_i)} - 1}{\omega_{\rho_i}^c - 1}. \qquad (6)
\end{aligned}$$

Now if ρ_i divides c, then each term of the sum in (5) equals one, so the sum is ρ_i; if ρ_i does not divide c, then one can see from (6) that the sum is zero. Hence, $\tilde{f}_c(\sigma)$ is one plus the sum of the ρ_i, where ρ_i divides c, which we showed was the bitrace of (σ, γ_c). □

Combining Theorem 34 with (4) yields a Frobenius formula for the rook partition algebra $RP_k(n+1)$.

Theorem 35 (*Frobenius formula*). *If μ is a partition of m with $0 \leqslant m \leqslant k$ and σ is any element of S_n, then*

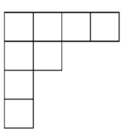
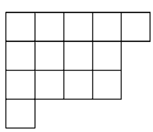
$$(n+1)^{k-m} \tilde{f}_\mu(\sigma) = \sum_{\substack{\lambda \vdash n \\ |\lambda^*| \leqslant k}} \chi_{S_n}^{\lambda}(\sigma) \chi_{RP_k(n+1)}^{\lambda}(d_\mu). \qquad (7)$$

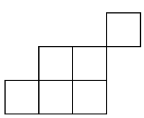
4.3. A Murnaghan–Nakayama rule for the rook partition algebra

The original Murnaghan–Nakayama rule is a recursive combinatorial formula used to compute character values of the symmetric group. We derive an analogous formula for the rook partition algebra using combinatorial objects similar to the ones used in the original rule. We begin by defining some of those combinatorial objects.

Definition 36. If μ and λ are two partitions such that $\mu \subseteq \lambda$, then the skew shape λ/μ is obtained by removing from λ the boxes contained in μ.

For example, if

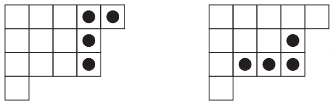
$\mu =$  and $\lambda =$  , then

$\lambda/\mu =$  .

Definition 37. A skew shape λ/μ is a border strip if it is connected and does not contain a 2×2 block of boxes. The size of a border strip is the number of boxes it contains.

Thus, our example above of a skew shape λ/μ is not a border strip.

Example 38. The following figures illustrate the 2 border strips of size 4 in $\lambda = (5, 4, 4, 1)$.



Definition 39. The height of a border strip λ/μ is the number of rows in the border strip minus one, and is denoted $ht(\lambda/\mu)$.

Thus, the first border strip in Example 38 has height 2, and the second border strip has height 1.

Definition 40. Suppose $\lambda \vdash n$ and $d \leqslant n$. A partition of $n-d$ that is contained in λ is denoted λ^{-d}.

Armed with these definitions, we are now ready to begin deriving our Murnaghan–Nakayama rule. Note that for each partition μ, the function $\tilde{f}_\mu$ is constant on conjugacy classes of S_n, and hence is a class function on S_n. Let $R(S_n)$ denote the $\mathbb{C}$-vector space of class functions on S_n. This vector space is equipped with an inner product $\langle , \rangle$ defined by

$$\langle f, g \rangle = \sum_{\rho \vdash n} \frac{1}{z_\rho} f(\gamma_\rho) g(\gamma_\rho),$$

where z_ρ is the size of the centralizer of ρ in S_n. The irreducible characters of S_n form an orthonormal basis of $R(S_n)$ with respect to this inner product. As a result, we have the following corollary of our Frobenius formula (7).

Corollary 41. *Let $\mu \vdash m$, where $0 \leqslant m \leqslant k$, and let $\lambda \vdash n$ such that $|\lambda^*| \leqslant k$. Then*

$$\chi^\lambda_{RP_k(n+1)}(d_\mu) = \langle (n+1)^{k-m} \tilde{f}_\mu, \chi^\lambda_{S_n} \rangle = (n+1)^{k-m} \langle \tilde{f}_\mu, \chi^\lambda_{S_n} \rangle.$$

We now consider the expansion of the product of class functions $\tilde{f}_c \chi^\delta_{S_n}$ in terms of the irreducible characters of S_n. This expansion is an rook partition algebra analogue of symmetric function identities for other centralizer algebras: the symmetric group (cf. I.3, ex. 11(2) in [11]), the Brauer algebra (Theorem 6.8 in [18]), the rook monoid algebra (Lemma 4.2 in [4]), and the partition algebra (Proposition 4.2.1 in [7]).

Lemma 42. *Let $\lambda, \delta \vdash n$ and let c be a positive integer. Then*

$$\langle \tilde{f}_c \chi^\delta_{S_n}, \chi^\lambda_{S_n} \rangle = \delta_{\delta,\lambda} + \sum_{d|c} \sum_{\substack{\lambda^{-d} \subseteq \lambda \\ \lambda^{-d} \subseteq \delta}} (-1)^{ht(\lambda/\lambda^{-d})} (-1)^{ht(\delta/\lambda^{-d})},$$

where $\delta_{\delta,\lambda}$ is 1 if $\delta = \lambda$ and 0 otherwise, and where the inner sum is over all partitions λ^{-d} of $n-d$ such that λ/λ^{-d} and δ/λ^{-d} are both border strips of size d. If, for a given d, no such λ^{-d} exists, the inner sum is zero.

Proof. Using the definition of the inner product,

$$\begin{aligned}\langle \tilde{f}_c \chi^{\delta}_{S_n}, \chi^{\lambda}_{S_n}\rangle &= \sum_{\alpha\vdash n} \frac{1}{z_\alpha} \tilde{f}_c(\alpha)\chi^{\delta}_{S_n}(\alpha)\chi^{\lambda}_{S_n}(\alpha)\\ &= \sum_{\alpha\vdash n} \frac{1}{z_\alpha}(1+f_c)(\alpha)\chi^{\delta}_{S_n}(\alpha)\chi^{\lambda}_{S_n}(\alpha)\\ &= \sum_{\alpha\vdash n} \frac{1}{z_\alpha}\chi^{\delta}_{S_n}(\alpha)\chi^{\lambda}_{S_n}(\alpha) + \sum_{\alpha\vdash n} \frac{1}{z_\alpha} f_c(\alpha)\chi^{\delta}_{S_n}(\alpha)\chi^{\lambda}_{S_n}(\alpha)\\ &= \langle \chi^{\delta}_{S_n}, \chi^{\lambda}_{S_n}\rangle + \sum_{\alpha\vdash n} \frac{1}{z_\alpha} f_c(\alpha)\chi^{\delta}_{S_n}(\alpha)\chi^{\lambda}_{S_n}(\alpha)\\ &= \delta_{\delta,\lambda} + \sum_{\alpha\vdash n} \frac{1}{z_\alpha} f_c(\alpha)\chi^{\delta}_{S_n}(\alpha)\chi^{\lambda}_{S_n}(\alpha),\end{aligned}$$

which, using the result from Proposition 4.2.1 of [7], equals

$$\delta_{\delta,\lambda} + \sum_{d|c} \sum_{\substack{\lambda^{-d}\subseteq\lambda\\ \lambda^{-d}\subseteq\delta}} (-1)^{ht(\lambda/\lambda^{-d})}(-1)^{ht(\delta/\lambda^{-d})},$$

where the inner sum is over all partitions λ^{-d} of $n-d$ such that λ/λ^{-d} and δ/λ^{-d} are both border strips of size d. $\square$

The following theorem is our analogue of the Murnaghan–Nakayama rule for the rook partition algebra.

Theorem 43. *Let $\lambda \vdash n$ with $|\lambda^*| \leqslant k$, where λ^* is the partition obtained from λ by removing its first part λ_1. Let μ be a partition with $0 \leqslant |\mu| \leqslant k$, and let $\bar{\mu}$ be the partition obtained from μ by removing a part of size c. Then*

$$\begin{aligned}\chi^{\lambda}_{RP_k(n+1)}(\gamma_\mu \otimes E^{\otimes k-|\mu|}) = {}& \chi^{\lambda}_{RP_{k-c}(n+1)}(\gamma_{\bar{\mu}} \otimes E^{\otimes k-c-|\bar{\mu}|})\\ &+ \sum_{d|c} \sum_{\substack{\delta=(\lambda^{-d})^{+d}\\ |\delta^*|\leqslant k-c}} \Big[(-1)^{ht(\lambda/\lambda^{-d})}(-1)^{ht(\delta/\lambda^{-d})}\\ &\times \chi^{\delta}_{RP_{k-c}(n+1)}(\gamma_{\bar{\mu}} \otimes E^{\otimes k-c-|\bar{\mu}|})\Big],\end{aligned}$$

where the outer sum is over all divisors d of c and the inner sum is over all $\delta \vdash n$ with $|\delta^*| \leqslant k-c$ such that δ is obtained from λ by removing a border strip of size d to obtain the partition λ^{-d} and then adding a border strip of size d to λ^{-d} to obtain $\delta = (\lambda^{-d})^{+d}$.

Proof. Using our Frobenius formula (7) for $\tilde{f}_{\bar{\mu}}$, we have that

$$(n+1)^{(k-c)-|\bar{\mu}|}\tilde{f}_{\bar{\mu}} = \sum_{\substack{\delta \vdash n \\ |\delta^*| \leqslant k-c}} \chi^{\delta}_{S_n} \chi^{\delta}_{RP_{k-c}(n+1)}(\gamma_{\bar{\mu}} \otimes E^{\otimes k-c-|\bar{\mu}|}). \tag{8}$$

Recall from Corollary 41 that

$$\chi^{\lambda}_{RP_k(n+1)}(\gamma_{\mu} \otimes E^{\otimes k-|\mu|}) = \langle (n+1)^{k-|\mu|}\tilde{f}_{\mu}, \chi^{\lambda}_{S_n}\rangle. \tag{9}$$

Since $\tilde{f}_{\mu} = \tilde{f}_{\bar{\mu}}\tilde{f}_c$ and $k-|\mu| = (k-c)-|\bar{\mu}|$, we can manipulate the right-hand side of (9) to get

$$\chi^{\lambda}_{RP_k(n+1)}(\gamma_{\mu} \otimes E^{\otimes k-|\mu|}) = \langle (n+1)^{(k-c)-|\bar{\mu}|}\tilde{f}_{\bar{\mu}}\tilde{f}_c, \chi^{\lambda}_{S_n}\rangle, \tag{10}$$

which equals

$$\left\langle \sum_{\substack{\delta \vdash n \\ |\delta^*| \leqslant k-c}} \chi^{\delta}_{RP_{k-c}(n+1)}(\gamma_{\bar{\mu}} \otimes E^{\otimes k-c-|\bar{\mu}|})\chi^{\delta}_{S_n}\tilde{f}_c, \chi^{\lambda}_{S_n} \right\rangle \tag{11}$$

using (8). So

$$\begin{aligned}
&\chi^{\lambda}_{RP_k(n+1)}(\gamma_{\mu} \otimes E^{\otimes k-|\mu|}) \\
&\quad = \left\langle \sum_{\substack{\delta \vdash n \\ |\delta^*| \leqslant k-c}} \chi^{\delta}_{RP_{k-c}(n+1)}(\gamma_{\bar{\mu}} \otimes E^{\otimes k-c-|\bar{\mu}|})\chi^{\delta}_{S_n}\tilde{f}_c, \chi^{\lambda}_{S_n} \right\rangle \\
&\quad = \sum_{\substack{\delta \vdash n \\ |\delta^*| \leqslant k-c}} \chi^{\delta}_{RP_{k-c}(n+1)}(\gamma_{\bar{\mu}} \otimes E^{\otimes k-c-|\bar{\mu}|})\langle \chi^{\delta}_{S_n}\tilde{f}_c, \chi^{\lambda}_{S_n}\rangle \\
&\quad = \sum_{\substack{\delta \vdash n \\ |\delta^*| \leqslant k-c}} \chi^{\delta}_{RP_{k-c}(n+1)}(\gamma_{\bar{\mu}} \otimes E^{\otimes k-c-|\bar{\mu}|}) \Bigg[\delta_{\delta,\lambda} \\
&\quad\quad + \sum_{d|c} \sum_{\substack{\lambda^{-d} \subseteq \lambda \\ \lambda^{-d} \subseteq \delta}} (-1)^{ht(\lambda/\lambda^{-d})}(-1)^{ht(\delta/\lambda^{-d})} \Bigg]
\end{aligned}$$

(by Lemma 42)

$$\begin{aligned}
&\quad = \chi^{\lambda}_{RP_{k-c}(n+1)}(\gamma_{\bar{\mu}} \otimes E^{\otimes k-c-|\bar{\mu}|}) \\
&\quad\quad + \sum_{\substack{\delta \vdash n \\ |\delta^*| \leqslant k-c}} \sum_{d|c} \sum_{\substack{\lambda^{-d} \subseteq \lambda \\ \lambda^{-d} \subseteq \delta}} \Big[(-1)^{ht(\lambda/\lambda^{-d})}(-1)^{ht(\delta/\lambda^{-d})} \\
&\quad\quad \times \chi^{\delta}_{RP_{k-c}(n+1)}(\gamma_{\bar{\mu}} \otimes E^{\otimes k-c-|\bar{\mu}|}) \Big],
\end{aligned}$$

where λ/λ^{-d} and δ/λ^{-d} are both border strips of length d. Any $\delta \vdash n$ that satisfies $\lambda^{-d} \subseteq \delta$, where also $\lambda^{-d} \subseteq \lambda$, must be able to be obtained by first removing a border strip of size d from λ (which gives λ^{-d}) and then adding a border strip of size d back on, which creates $\delta = (\lambda^{-d})^{+d}$. In other words, we can rewrite the triple sum above to obtain our desired result, namely

$$\begin{aligned}\chi^{\lambda}_{RP_k(n+1)}(\gamma_\mu \otimes E^{\otimes k-|\mu|}) = {} & \chi^{\lambda}_{RP_{k-c}(n+1)}(\gamma_{\bar{\mu}} \otimes E^{\otimes k-c-|\bar{\mu}|}) \\ & + \sum_{d|c} \sum_{\substack{\delta=(\lambda^{-d})^{+d} \\ |\delta^*| \leqslant k-c}} \Big[(-1)^{ht(\lambda/\lambda^{-d})}(-1)^{ht(\delta/\lambda^{-d})} \\ & \times \chi^{\delta}_{RP_{k-c}(n+1)}(\gamma_{\bar{\mu}} \otimes E^{\otimes k-c-|\bar{\mu}|})\Big],\end{aligned}$$

where the inner sum is over all $\delta \vdash n$ with $|\delta^*| \leqslant k-c$ such that δ is obtained from λ by removing a border strip of size d to obtain the partition λ^{-d} and then adding a border strip of size d to λ^{-d} to obtain $\delta = (\lambda^{-d})^{+d}$. $\square$

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